The Maxwell wave function of the photon

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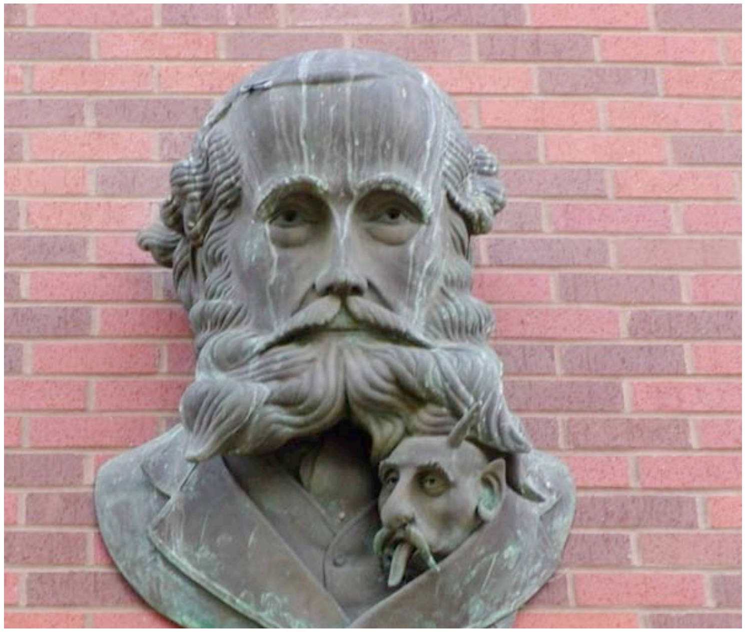
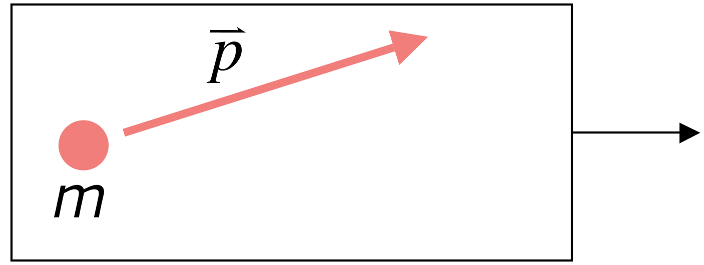
James Clerk Maxwell
(1864)

"This velocity is so nearly that of light that it seems we have strong reason to conclude that light itself is an electro-magnetic disturbance in the form of waves propagated through the electro-magnetic field."



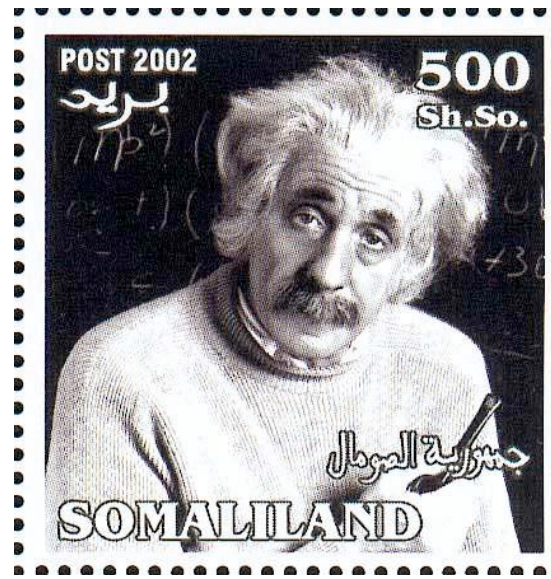
Albert Einstein

Special Relativity (1905)



- the laws of physics look the same in any moving frame.
- light travels at the same speed *c* in any frame.
- the energy of a particle with rest mass *m* , moving with momentum *p* is:

$$E = \sqrt{(mc^2)^2 + (cp)^2}$$



- for slow particles:

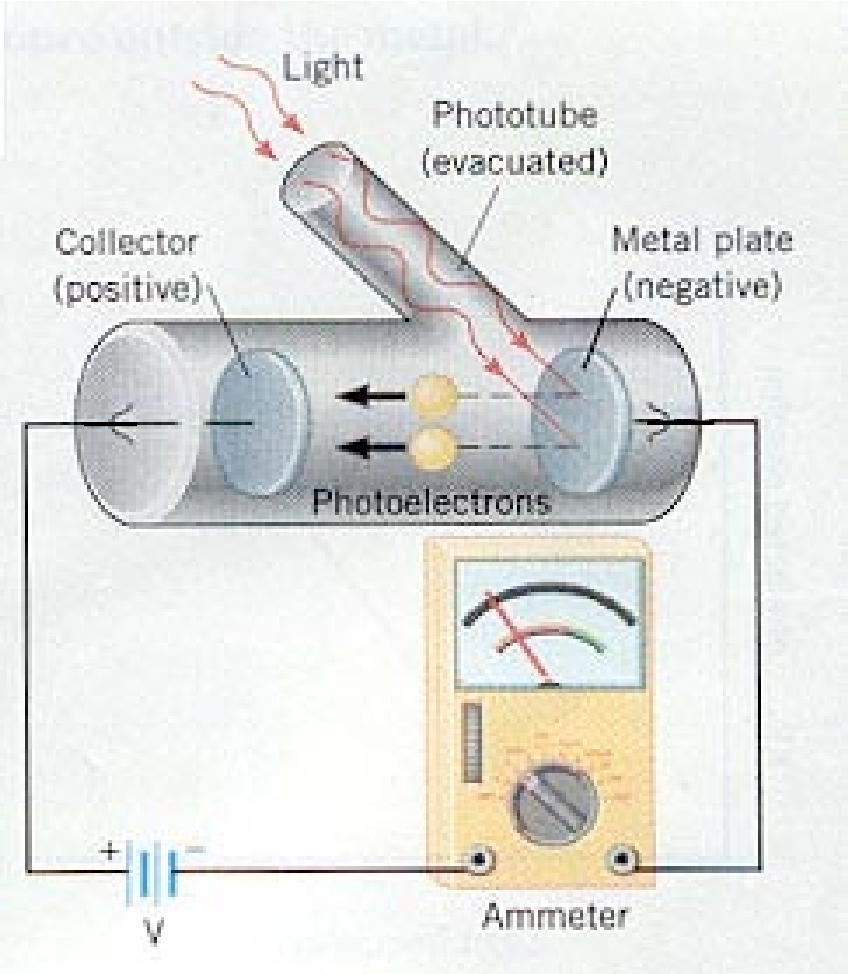
$$E \simeq mc^2 + \frac{p^2}{2m} + \ldots$$

Einstein rest energy

Newton kinetic energy

W. Hallwachs and P. Lenard
Photoelectric effect (1900) -

- *Einstein* hypothesized that light is made of ***particle-like*** objects called quanta, and later "photons." Each photon has energy $E = h\nu$.



Einstein (1924): "*There are therefore now two theories of light, both indispensable, and - as one must admit today despite twenty years of tremendous effort on the part of theoretical physicists - without any logical connection.*"

the modern view:

a photon is an excitation (that is, a state) of a quantum field.

still:

it is tempting to think of a photon as a kind of particle.

question:

if a photon is a particle, what is its quantum wave function, and what wave equation does it obey?

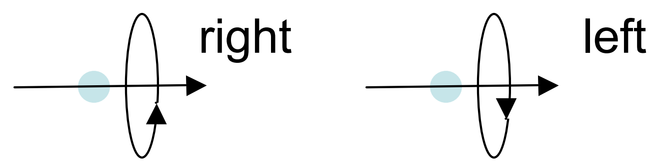
What are the differences between electrons and photons?

Electron has:

- nonzero mass
- any speed < c
- Spin = 1/2 (two possible projections along any chosen axis, +1/2, -1/2)
- -> two-component *wave function* $\Psi^{(2)}$

  obeying the Schrödinger equation: $$i\hbar \frac{\partial}{\partial t} \Psi^{(2)} \cong -\frac{\hbar^2}{2m} \vec{\nabla}^2 \Psi^{(2)}$$

Photon has:



- zero mass
- speed = c
- Spin = 1 (two possible projections along propagation axis, +1, -1)
- *wave function* obeying what equation? how many components?

Paul Dirac (Cambridge, 1928)

-relativistic quantum theory of electron

$$E = \sqrt{(mc^2)^2 + (cp)^2}$$

m=mass

p=momentum

m≠O

s=1/2

$$i\hbar \frac{\partial}{\partial t}\Psi = \sqrt{(mc^2)^2 + c^2(-i\hbar\vec{\nabla})^2}\;\Psi$$

Dirac Equation

$$i\hbar \frac{\partial}{\partial t}\Psi = cm\underline{\underline{\beta}}\Psi - i\hbar c(\underline{\underline{\vec{\alpha}}}\bullet\vec{\nabla})\Psi$$

v<<c

Schrödinger Equation

$$i\hbar \frac{\partial}{\partial t}\Psi^{(2)} \cong -\frac{\hbar^2}{2m}\vec{\nabla}^2\Psi^{(2)}$$

2 components

(spin up, down)

Derivation of Maxwell's Equations

Photon, m=0,

s=1, 3 components

momentum wave fn

$$\tilde{\psi}(\vec{p},E) = (\tilde{\psi}_x, \tilde{\psi}_y, \tilde{\psi}_z)$$

$$\tilde{\psi} = \tilde{\psi}_T + \tilde{\psi}_L$$

$$\vec{p} \times \tilde{\psi}_L = 0, \quad \vec{p} \cdot \tilde{\psi}_T = 0$$

$$E = \sqrt{(c\,p)^2}$$

$$E\,\tilde{\psi}(\vec{p},E) = c\sqrt{\vec{p}\cdot\vec{p}}\;\tilde{\psi}(\vec{p},E)$$

$$\hat{O} \doteq \sqrt{\vec{p}\cdot\vec{p}} \overset{?}{=} i\,\vec{p}\times$$

$$\hat{O}\hat{O}\tilde{\psi}_T = i\vec{p}\times(i\vec{p}\times\tilde{\psi}_T) = (\vec{p}\cdot\vec{p})\tilde{\psi}_T - \vec{p}(\vec{p}\cdot\tilde{\psi}_T) = (\vec{p}\cdot\vec{p})\tilde{\psi}_T$$

$$\boxed{E\,\tilde{\psi}_T(\vec{p},E) = c\,i\,\vec{p}\times\tilde{\psi}_T(\vec{p},E)}$$

weight

$$\vec{\psi}_T(\vec{r},t) = \iint dE\,d^3p\;\; \delta(E - c|\vec{p}|)\exp(-iEt/\hbar + i\vec{p}\cdot\vec{r}/\hbar)\,f(E)\,\tilde{\psi}_T(\vec{p},E)$$

$$\boxed{i\frac{\partial}{\partial t}\vec{\psi}_T(\vec{r},t) = c\,\vec{\nabla}\times\vec{\psi}_T(\vec{r},t)}$$

$$\vec{\psi}_T(\vec{r},t) = \vec{E} + i\vec{B} \left\{ \begin{array}{l} \dfrac{\partial}{\partial t}\vec{B} = -c\,\vec{\nabla}\times\vec{E} \\[2ex] \dfrac{\partial}{\partial t}\vec{E} = c\,\vec{\nabla}\times\vec{B} \end{array} \right.$$

Riemann-Silberstein vector

For a single-photon field, the complex electromagnetic field (*E*+*iB*) is the quantum wave function of the photon.

$$\psi(\vec{r}) = \begin{pmatrix} E_x(\vec{r}) + iB_x(\vec{r}) \\ E_y(\vec{r}) + iB_y(\vec{r}) \\ E_z(\vec{r}) + iB_z(\vec{r}) \end{pmatrix}$$

I. Bialynicki-Birula (Ustron1993)
J. Sipe (PRA 1995)
M.Scully M.Zubairy (1997)

Maxwell, in 1862, discovered a relativistic, quantum mechanical theory of a single photon.

added after the talk:

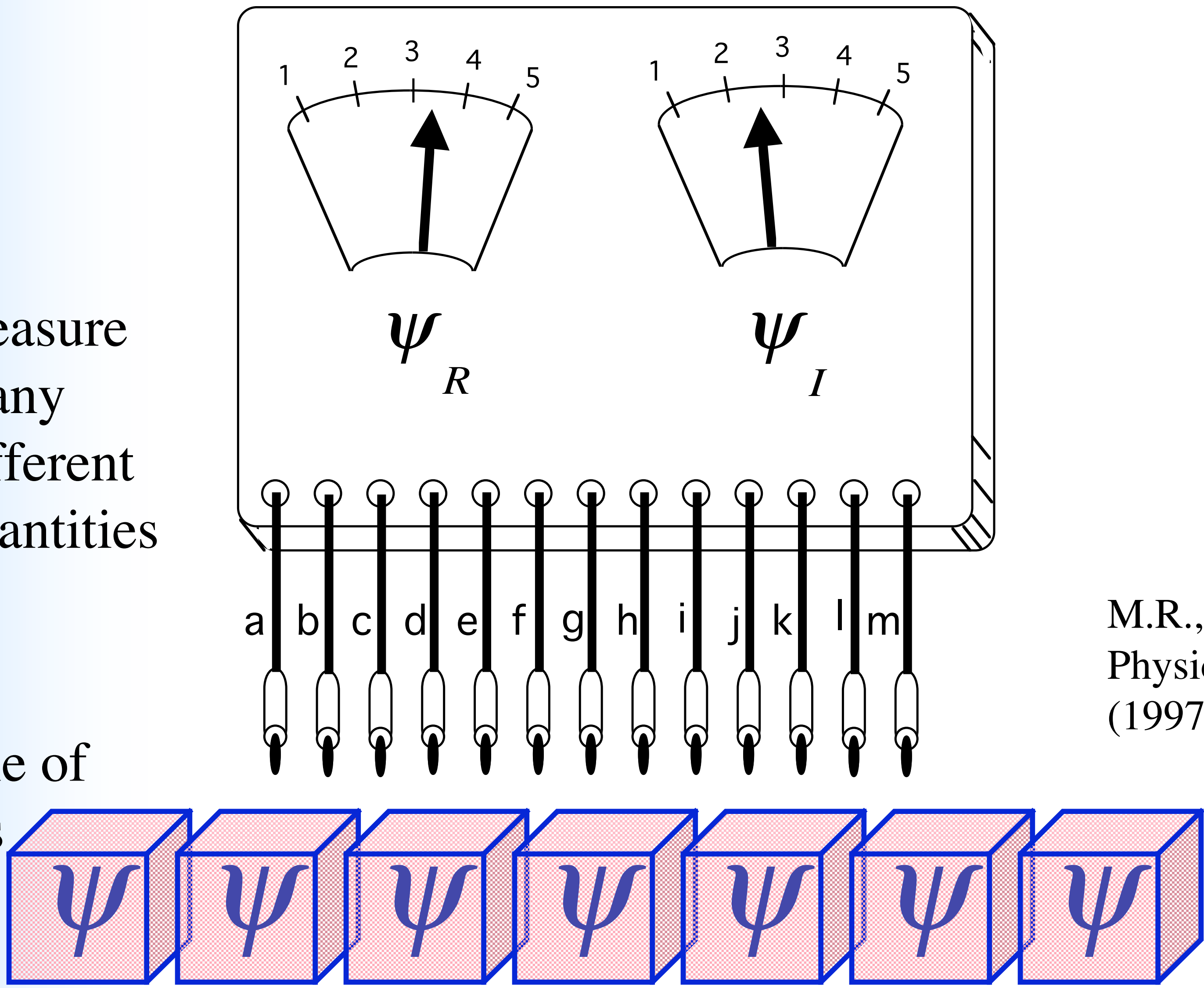
1. Scully's student talk mentioned adding higher multipoles (but did not say it included magnetic poles, but it could).

2. need to comment that psi^2 is not the prob. to find the photon at r, rather to find the energy at r.

3. even massive particles have this distinction with mass, charge, when relativistic.

4. the Klein gordon eqn comes from E^2=p^2. scalar wv eqn, no vector properties.

QUANTUM STATE TOMOGRAPHY



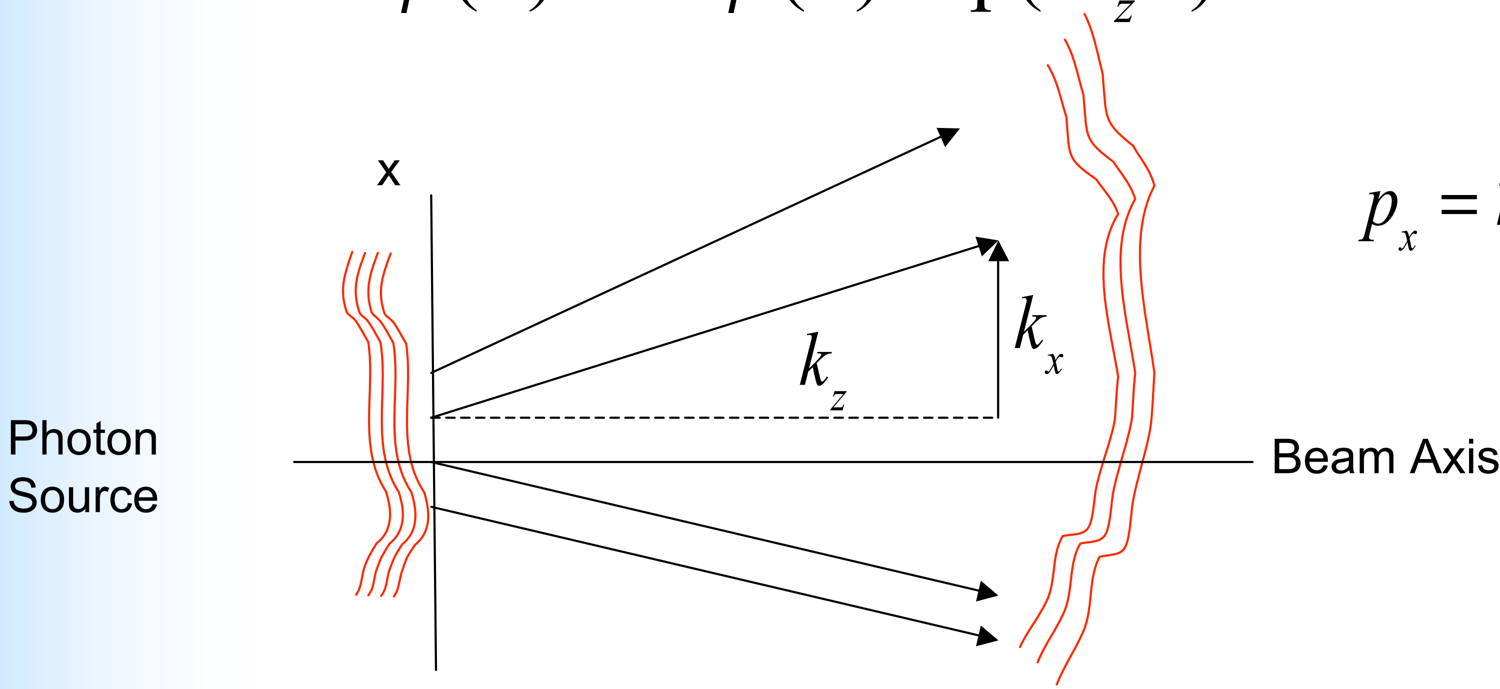
measure many different quantities

ensemble of particles

M.R., Contemp. Physics **38**, 343 (1997)

Wave-function Fronts (paraxial)

$$\overrightarrow{\psi}(x) = \overrightarrow{\varepsilon}\,\psi(x)\exp(ik_z z)$$



$$p_x = \hbar k_x$$

Near Field

Far-Field Diffraction

Wigner Function

a Quasi-joint probability function for x, k_x

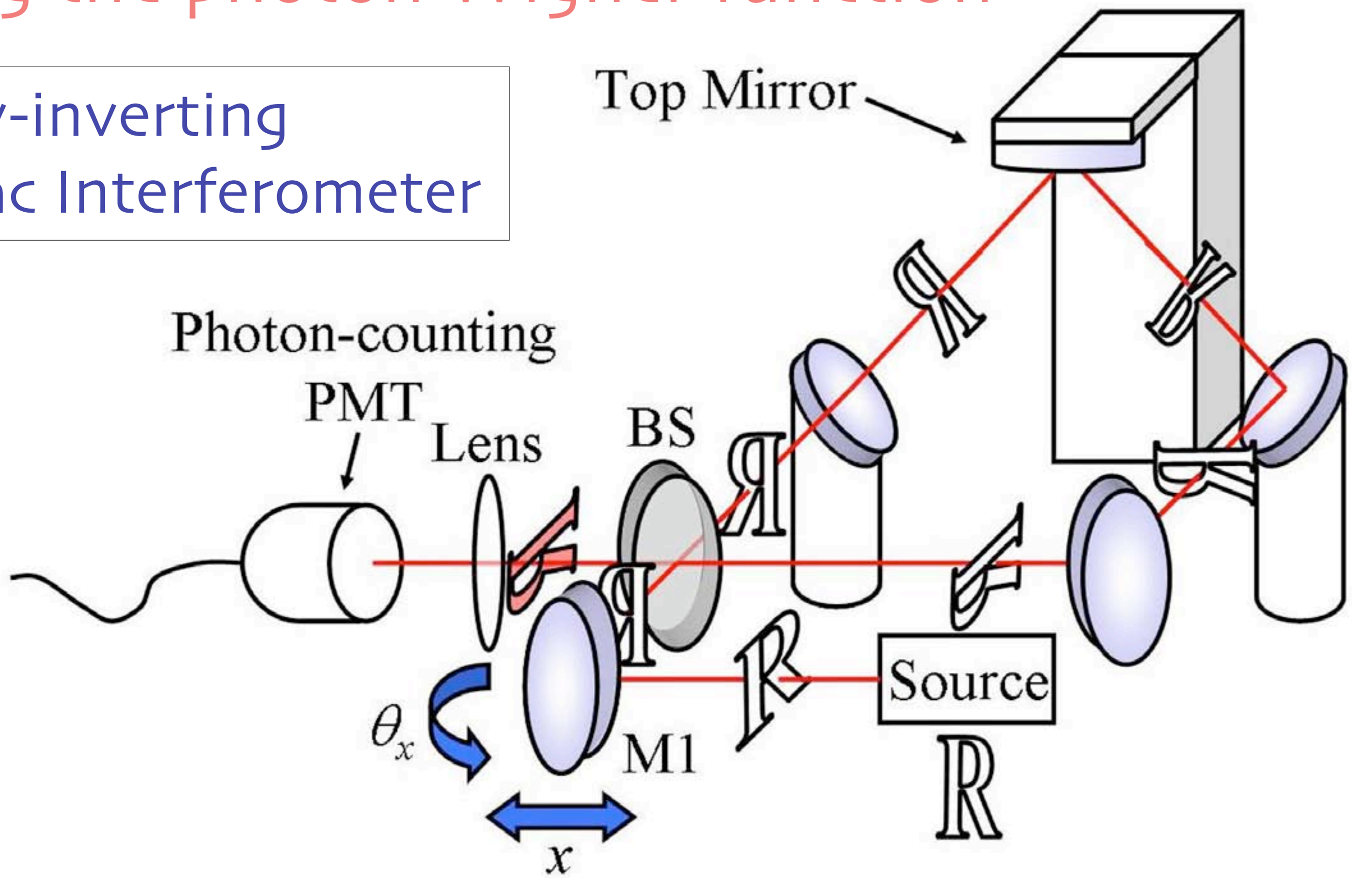
$$W(p_x, x) = \int \psi(x + x')\psi^*(x - x')e^{-2i p_x x'/\hbar}\, dx'$$

The Wigner Function is uniquely related to the Wave Function, so its measurement reveals $\psi(x)$ not just $|\psi(x)|^2$

for photon: $\psi(x) \simeq \vec{\varepsilon}\left(E_x(x) + iB_x(x)\right)$

Measuring the photon Wigner function

Parity-inverting
Sagnac Interferometer



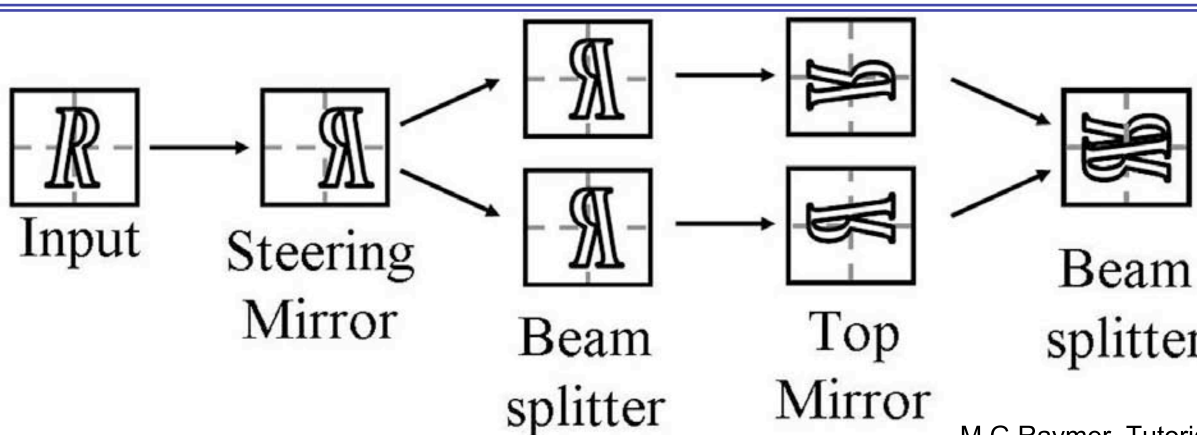
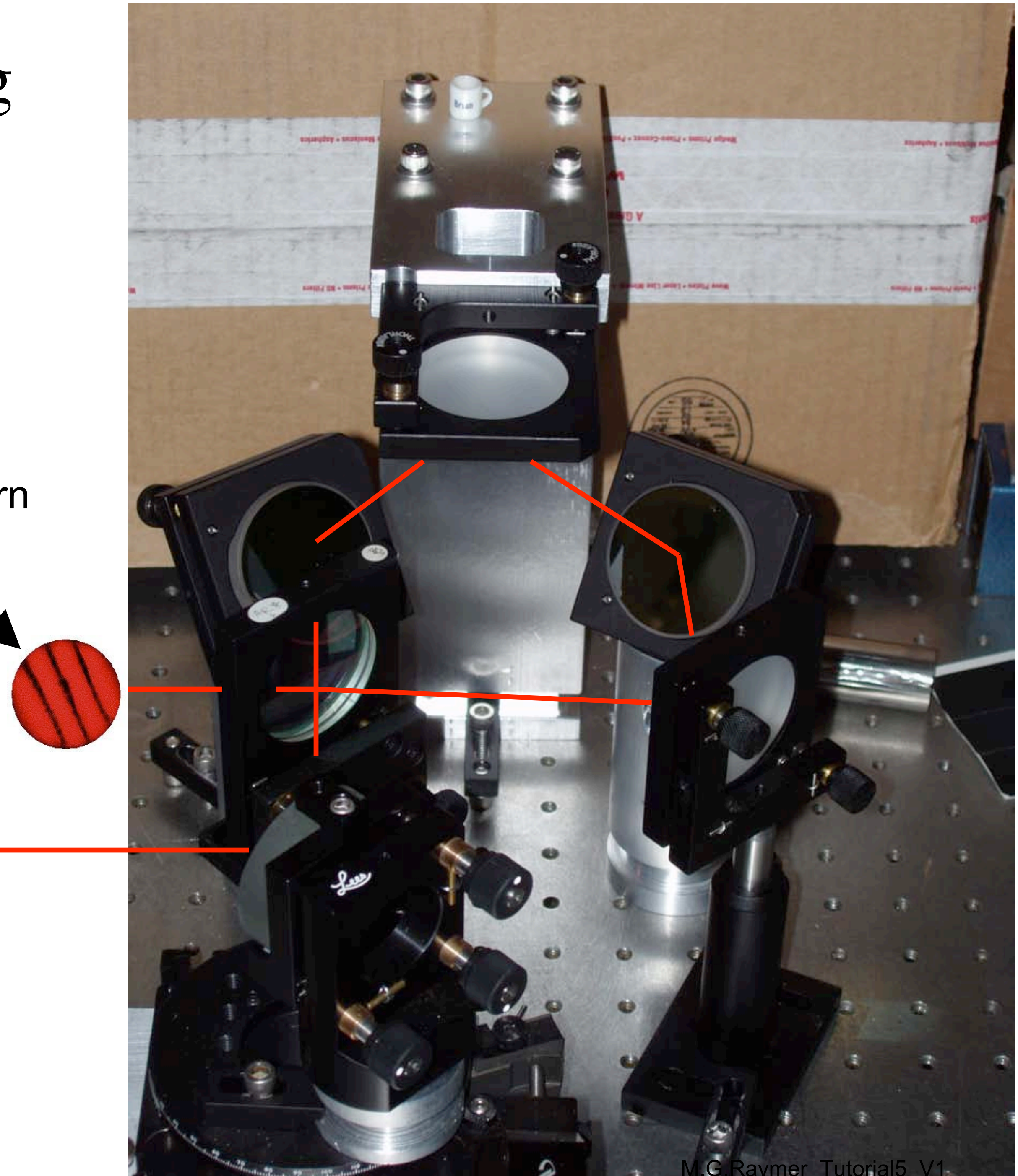


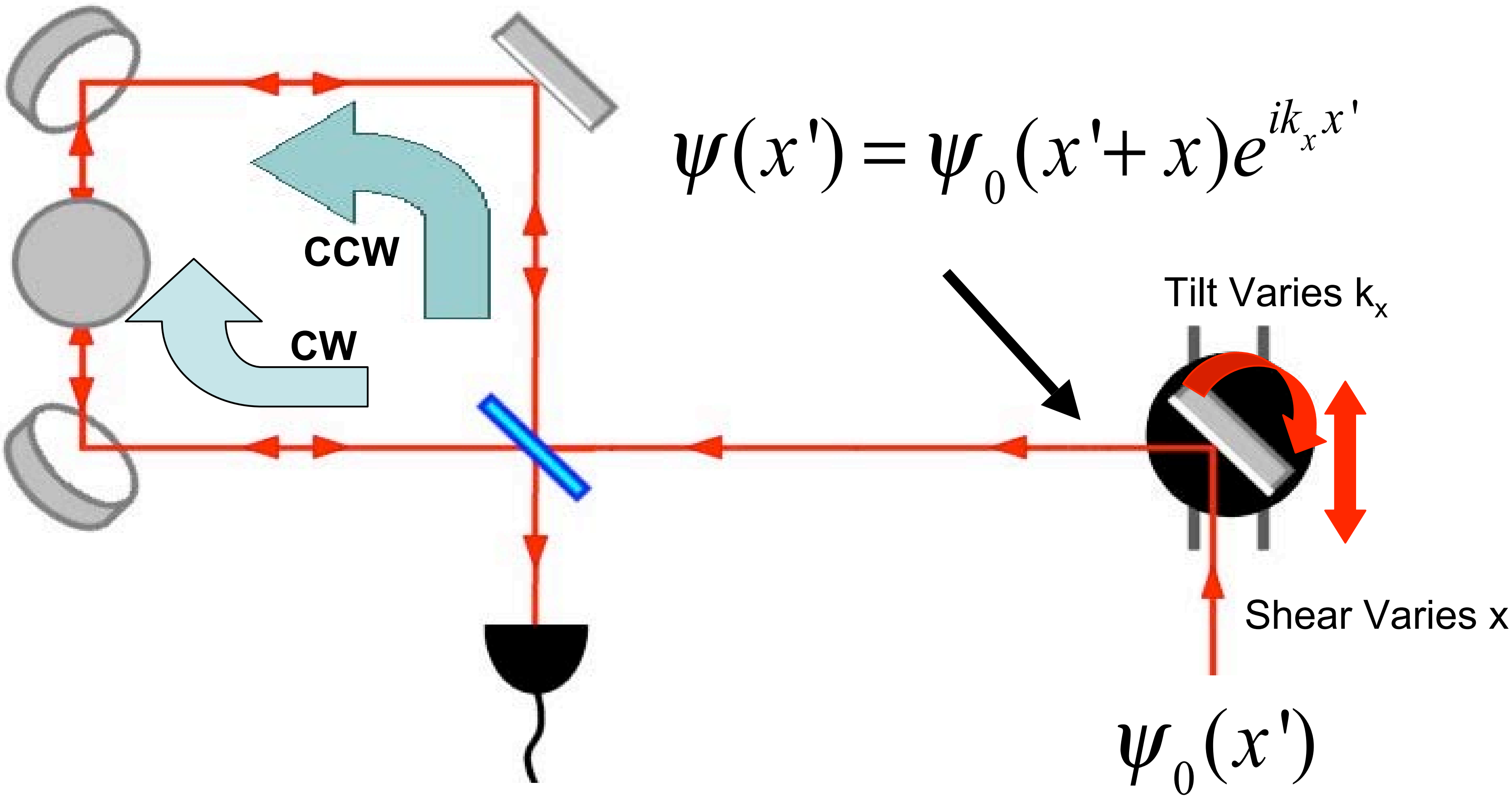
Diagram by Brian Smith

Parity-Inverting Sagnac Interferometer

Interference Pattern



Measuring the Wigner Function of Photons with a parity-inverting Sagnac interferometer



photomultiplier (detects single photons)

Diagram by Bryan Killet

Output Field

$$E_{Output}(x') = E_{CW}(x') + E_{CCW}(x')$$

$$E_{Output}(x') = \frac{1}{\sqrt{2}}\{E_0(-x'+x)e^{-ik_x x'} + E_0(x'+x)e^{ik_x x'}\}$$

The average photon count rate equals the intensity integrated over the whole detector surface.

$$I \propto \int \left|E_{output}(x')\right|^2 dx' \cong C_1 + C_2 \cdot \int_{-\infty}^{+\infty} E_0(x+x')E_0^{\,*}(x-x')e^{-2ik_x x'}dx'$$

Direct measurement of Wigner Function

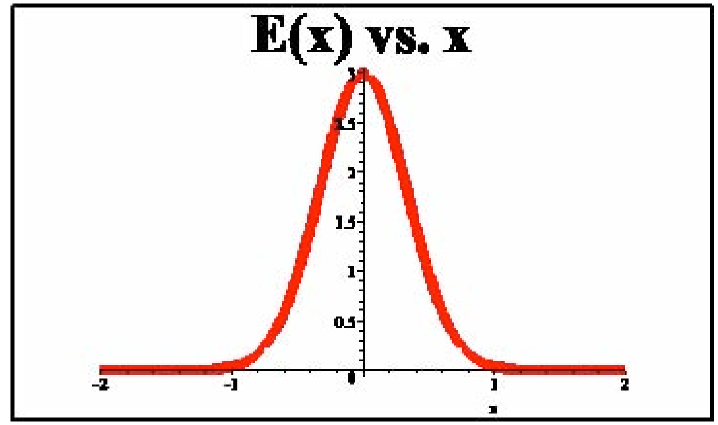
The average photon count rate equals:

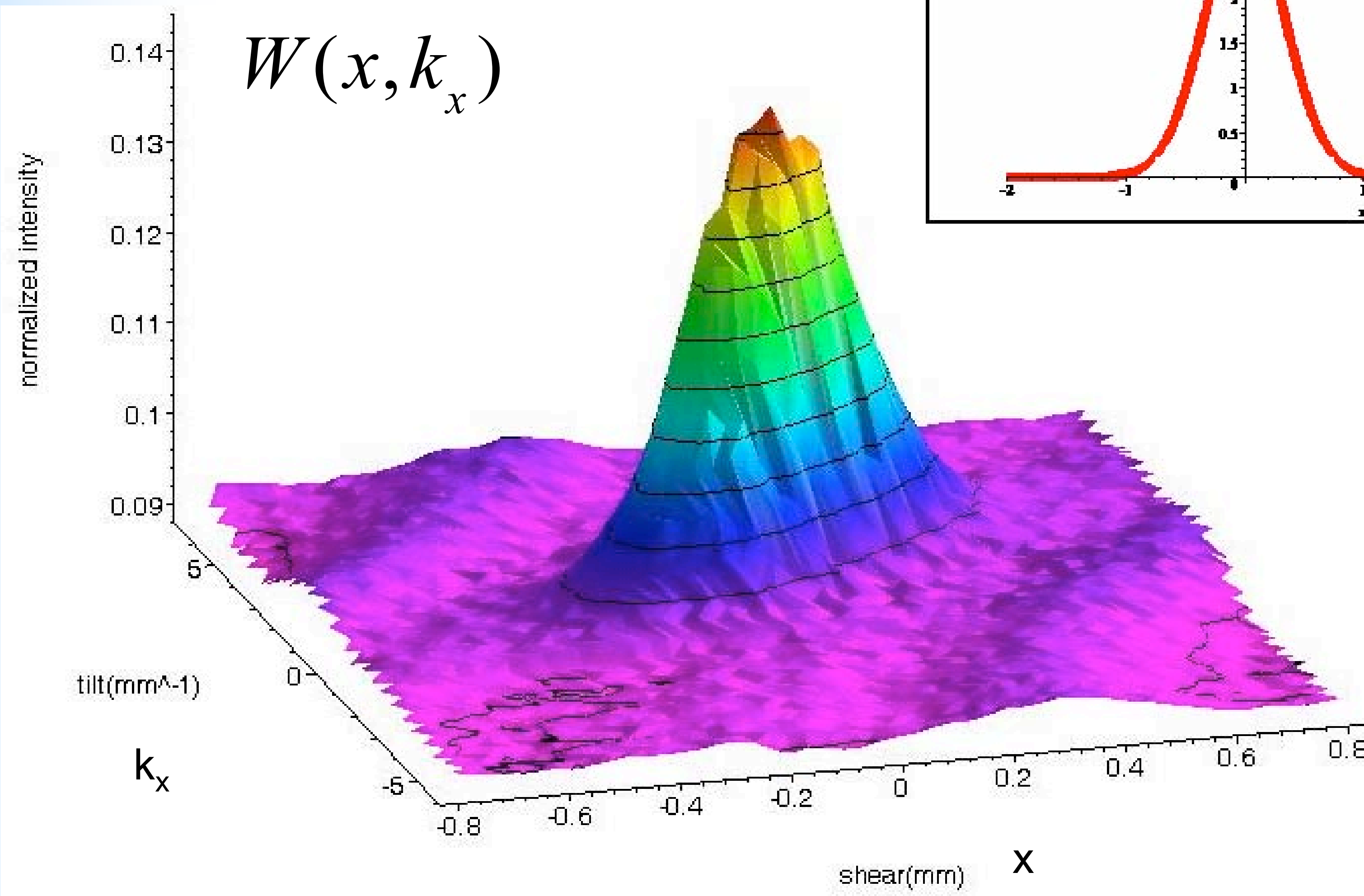
$$\underbrace{I \cong C_1 + C_2 \cdot \int_{-\infty}^{+\infty} \psi_0(x+x')\psi_0{}^*(x-x')e^{-2ik_x x'}dx'}$$

$$I \cong \mathrm{C}_1 + \mathrm{C}_2 \cdot W(x, k_x)$$

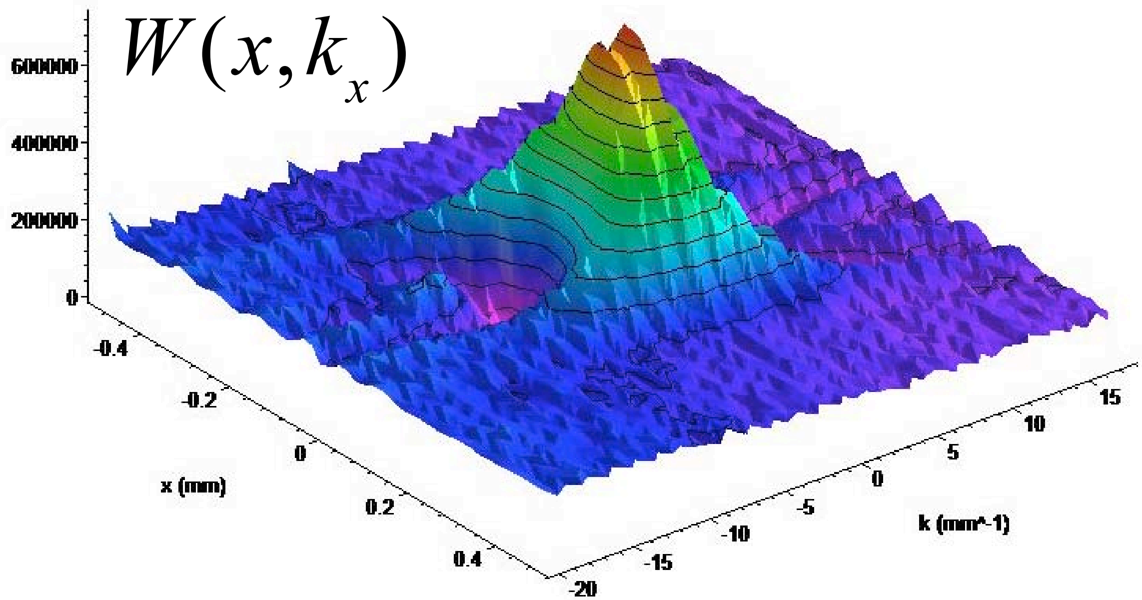
Results – Gaussian Beam

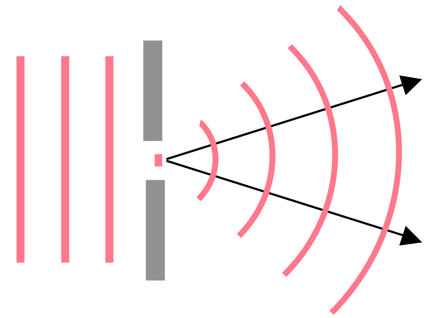
photodiode detector

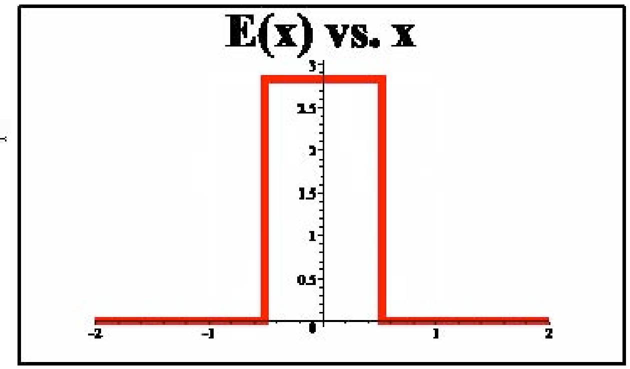




Single Slit - photon-counting detector



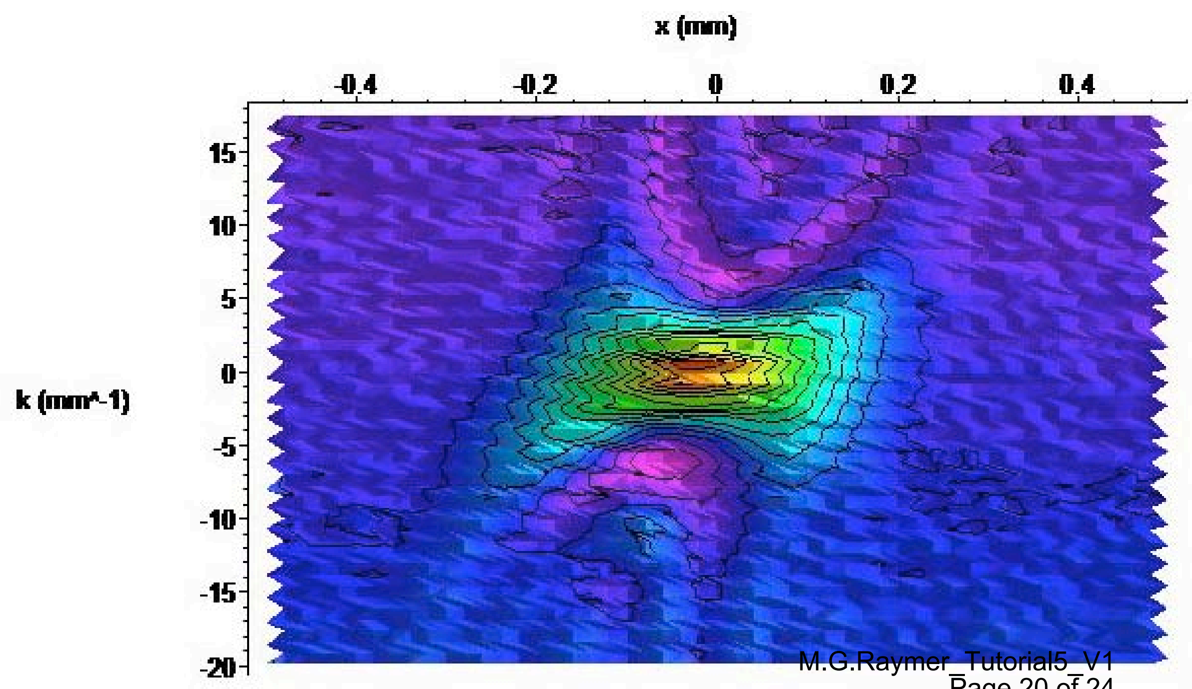




Measured Wigner function for ensemble of single photons.

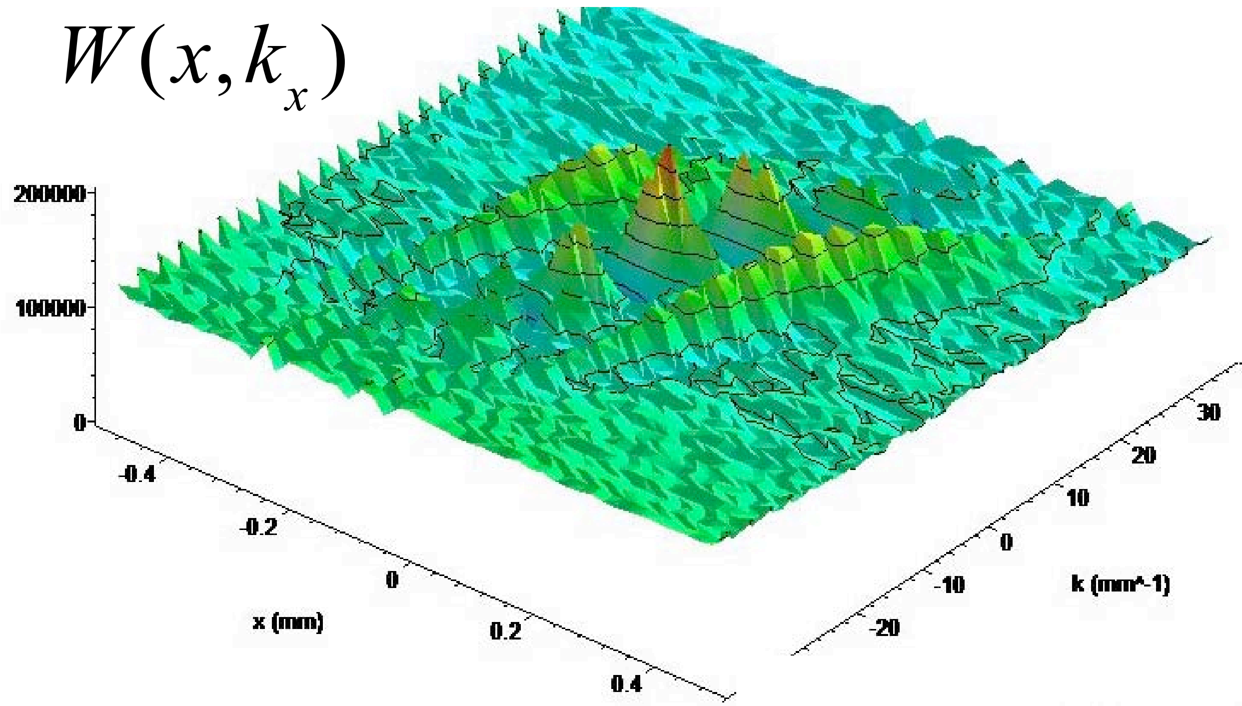
x = position

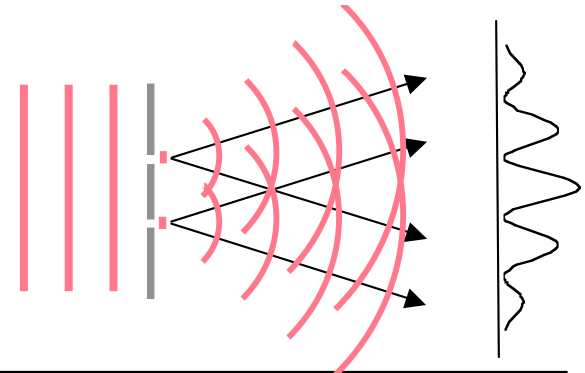
k_x = transverse momentum

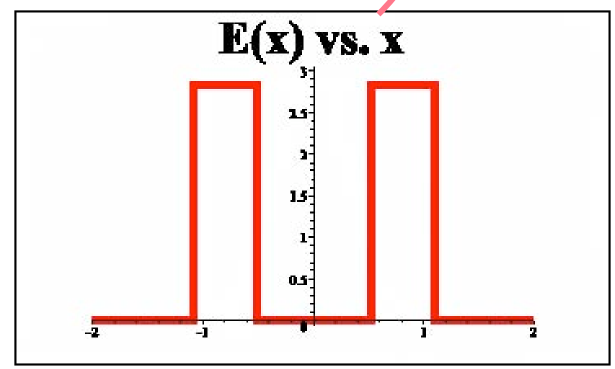


Double Slit - photon-counting detector

$$W(x,k_x)$$







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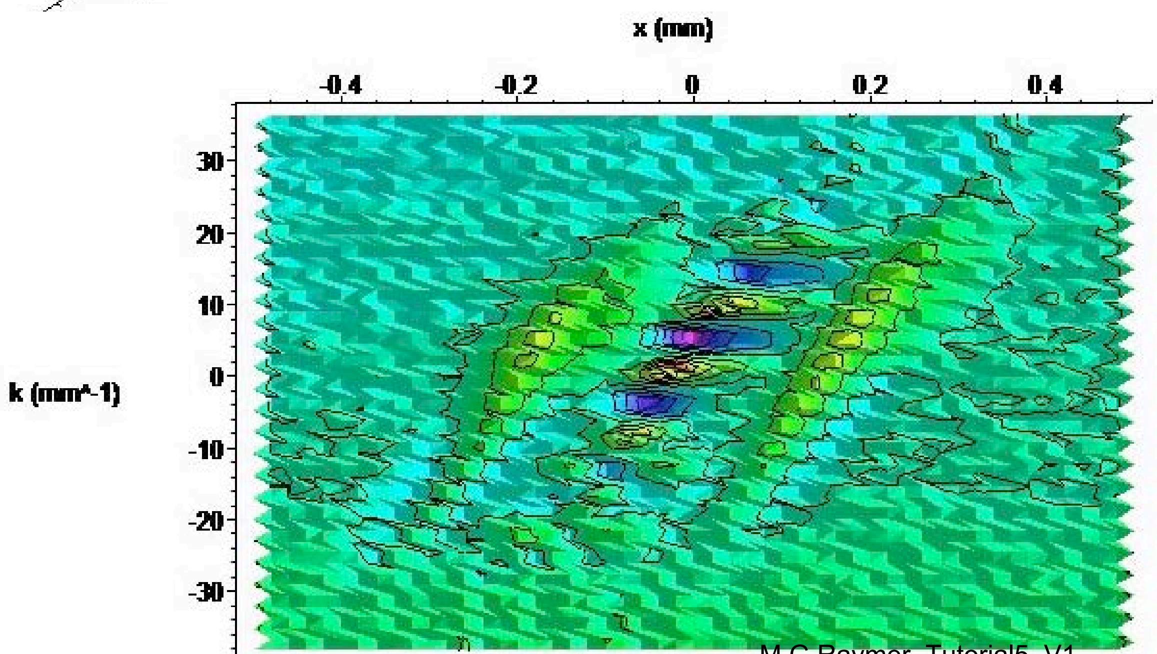


Two-photon Maxwell's equations
- two possible forms

one-time:

$$\boxed{\frac{i}{c}\frac{\partial}{\partial t}\vec{\psi}(\vec{r}_1,\vec{r}_2,\underline{t}) = \vec{\nabla}_1 \times \vec{\psi}(\vec{r}_1,\vec{r}_2,\underline{t}) + \vec{\nabla}_2 \times \vec{\psi}(\vec{r}_1,\vec{r}_2,\underline{t})}$$

two-time:

$$\frac{i}{c}\frac{\partial}{\partial t_1}\vec{\psi}(\vec{r}_1,\underline{t_1};\vec{r}_2,\underline{t_2}) = \vec{\nabla}_1 \times \vec{\psi}(\vec{r}_1,\underline{t_1};\vec{r}_2,\underline{t_2})$$

$$\frac{i}{c}\frac{\partial}{\partial t_2}\vec{\psi}(\vec{r}_1,\underline{t_1};\vec{r}_2,\underline{t_2}) = \vec{\nabla}_2 \times \vec{\psi}(\vec{r}_1,\underline{t_1};\vec{r}_2,\underline{t_2})$$

These are equivalent under the measurement-collapse hypothesis.

one-time Max.Eqn.

$$\boxed{\frac{i}{c}\frac{\partial}{\partial t}\vec{\psi}(\vec{r}_1,\vec{r}_2,t)=\vec{\nabla}_1\times\vec{\psi}(\vec{r}_1,\vec{r}_2,t)+\vec{\nabla}_2\times\vec{\psi}(\vec{r}_1,\vec{r}_2,t)}$$

solution:

$$\boxed{\vec{\psi}(\vec{r}_1,\vec{r}_2,t)=\sum_j C_j\,\vec{\psi}_j(\vec{r}_1,t)\otimes\vec{\phi}_j(\vec{r}_2,t)}$$

at time T_1, measure $\vec{r}_1$, obtain value $\vec{R}_1$

$$\boxed{\vec{\psi}(\vec{r}_1,\vec{r}_2,t)\Rightarrow\vec{\psi}(\vec{R}_1,\vec{r}_2,t_2)=\sum_j\left[C_j\,\vec{\psi}_j(\vec{R}_1,T_1)\right]\otimes\vec{\phi}_j(\vec{r}_2,t_2)}$$

same form as two-time wave function before measurement:

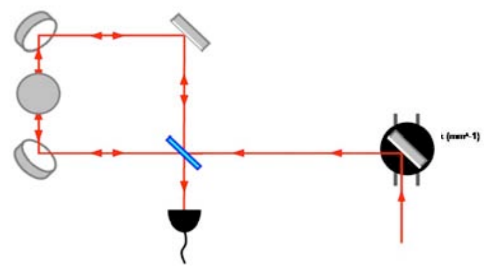
$$\boxed{\vec{\psi}(\vec{r}_1,t_1;\vec{r}_2,t_2)=\sum_j C_j\,\vec{\psi}_j(\vec{r}_1,t_1)\otimes\vec{\phi}_j(\vec{r}_2,t_2)}$$

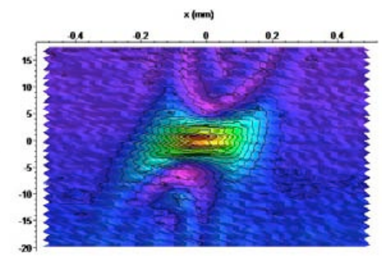
NSF

Synopsis

- Particle kinematics->Single-photon Maxwell Equation for quantum wave function.

- Direct measurement of Wigner distribution for transverse sta of photon.





- Two-photon wave function.

- Application: characterization of quantum channels and quantum logic gates